Decoherence: A View from Topology

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Abstract

This paper introduces a topological approach to decoherence that can be seen as an extension of the consistent histories approach to quantum mechanics. The approach uses the formal tools of categorical algebraic topology and sheaf theory to capture the relationship between a global description of a quantum system in terms non-commutative algebras of quantum observables and a local description in terms of local commutative algebras associated with particular measurement contexts. We claim that the difference between the algebraic structure of quantum observables and the algebraic structure of quasi-classical observables at suitable coarse-grained scales is essentially ignored in the consistent histories approach to decoherence, and that the notorious problems of the approach can be overcome by an appropriate topological treatment of this difference. We describe decoherence as a process, in which a global quantum description is reduced to local quasi-classical descriptions according to specific topological compatibility conditions.

Keywords: Consistent Histories, Decoherence, Sheaf Theory, Topology

1. Introduction

It is commonly acknowledged that decoherence by itself provides neither an interpretation of quantum mechanics nor a solution to the measurement problem, i.e. the question of why the dynamics of a quantum system is correctly described by a unitary Schrödinger equation as long as no measurements are performed, and why this description fails when the state of the system is determined in a measurement. Although decoherence has become part of the "new orthodoxy" (Bub 1997) in quantum mechanics and although the phenomenon of decoherence has been well established in the laboratory over the last decades, there exists currently no generic physical model for decoherence and the question remains open whether there can even be such a unique model model or whether we need several models each accounting for different sources of decoherence.[1] Definitely, decoherence is itself in need of a feasible interpretation. In the best case, an adequate understanding of decoherence might lead the way towards a solution of the measurement problem. This paper proposes a novel topological approach to decoherence based on category and sheaf-theoretic ideas and methods, used previously for the analysis of quantum event algebras and quantum observable algebras

[1]For recent developments especially regarding experiments on decoherence in large systems and low-energy physics, as well as for the question of a unified model of decoherence see Stamp (2006).

(Zafiris 2004, 2006a, 2007. For a general introduction to category theory and sheaf theory see MacLane and Moerdijk 1992; MacLane 1998).

For the purpose of what follows, it is important to highlight the differences between the two current major approaches to decoherence: Environment-induced decoherence on the one hand, and the consistent (or decoherent) histories approach on the other. Environment-induced decoherence (often referred to as the 'decoherence program"; for comprehensive reviews including relevant literature see Bacciagaluppi 2012; Schlosshauer 2005, 2007; Zurek 2003) studies correlations between the states of a quantum system and the states of its environment and, in particular, the local suppression of quantum interference through the interaction with the environment. Upon interaction with the environment, a quantum system (in its pure state) becomes entangled with the latter resulting a global superposition state involving the system as well as the environment. Averaging over the large degrees of freedom in the environment allows for a local vanishing of interference terms yielding a mixed state for the system. Thus understood, decoherence is also supposed to account for the fact that, upon measurement, we never find a system in its pure state; that is to say, we never directly observe superpositions. Note that suppression of interference is only *local* and that global coherence is retained in the global system-environment complex.

Usually, the interaction between a system and its environment is thought to involve a transfer of energy. This energy transfer is theoretically accounted for via an effective Hamiltonian, as it is traditionally done, for example, in oscillator bath models of decoherence (see Feynman and Vernon 1963; Caldeira and Leggett 1983; Leggett 1984; Leggett et al. 1987; all references taken from Stamp 2006, p. 470). Recent experiments show, however, that decoherence effects can also be observed in non-dissipative models (Bertaina et al., 2008). Thus the conventional description of system-environment interactions via an effective Hamiltonian is arguably incomplete. It can therefore be assumed that non-dissipative decoherence cannot be simply reduced to the dynamics of causal interactions involving an effective Hamiltonian, but that it is related to an exchange of phase between the local system and its environment.[2] The question has been raised whether there are "intrinsic" sources of decoherence in nature – sources that are not reducible to dissipative processes. By "intrinsic" is meant an inevitable decoherence that perhaps arises as part of the basic structure of the universe that would even be operative at T=0 (Stamp 2006, p. 490; intrinsic decoherence in low energy physics was suggested by Mohanty et al. 1997).

A further problem of the environment induced approach to decoherence is related to the decomposition of the universe, described by a global state vector, into "system",

[2]For a non-dissipative spin-bath model of "precessional decoherence" and a model of "third-party decoherence" see Stamp (2006) and references therein. Stamp notes "[...] that decoherence caused by a quantum environment is really about phase exchange between system and environment, and has no necessary connection with either environmental noise or dissipation at all", p. 482. Schlosshauer (2007) also writes that "decoherence may, but does not have to be accompanied by dissipation[...]", p. 93.

"apparatus" and "remaining environment" with their respective state vectors. This decomposition into subsystems is highly nontrivial (see Lombardi et al. 2012; see also Zurek 1998, p. 1820). The second major approach to decoherence, the consistent histories approach, tries to bypass the decomposition problem by focussing on histories of the whole universe itself (see Griffiths 1984, 1993; Gell-Mann and Hartle 1990, 1993; Omnès 1988; for reviews of the program with further literature see Bassi and Ghirardi 2000; Griffiths 2002; Halliwell 1995; Omnès 1992, 2002; Schlosshauer 2005). In this perspective, decoherence typically arises through the vanishing of the off-diagonal terms of the so called decoherence functional for appropriate families of coarse-grained histories. The major problem with this approach is that there exist many sets of consistent histories which cannot be combined to yield a maximal consistent description, and that there is no criterion of unique selection among them. It has been recognized that the consistent histories approach can be considered as an interesting realistic alternative to the orthodox interpretation of quantum mechanics only if a precise criterion for the characterization of appropriate families of decoherent histories can be given (see Bassi and Ghirardi 2000, p. 1).

In the following, we propose topological approach that can be seen as an extension of the consistent histories approach. We believe that the problem of providing an adequate criterion for consistency can be bypassed if decoherence is depicted as a genuinely topological phenomenon that involves the reduction of a global non-Boolean quantum algebra to coarse-grained local descriptions in terms of quasi-classical local Boolean algebras.

The link between the consistent histories approach and the topological approach to decoherence is established in two steps: First, by explicitly considering the difference in the algebraic structure between an algebra of microscopic (i.e. quantum) observables and various algebras of macroscopic (i.e. quasi-classical) observables at suitable coarse-grained scales with respect to their commutativity; and second, by topologizing this difference in terms of a local-to-global sheaf-theoretic construction. Via the latter, the information of a global, non-commutative algebra can be a) localized by restriction to compatible families of local commutative algebras of classical microscopic observables, and, inversely, b) globalized by extension via the topological method of gluing, thereby recovering its content (see Zafiris 2004, 2006, 2007). The proposed topological approach is based on fundamental topological representation theorems of modern mathematics (Stone theorem, Gelfand theorem and Grothendieck theorem), aiming at the isomorphic representation of algebras of observables by sections of sheaves over appropriate topological spectrum spaces.

Note that the approach does not suggest any changes to quantum theory; in particular it does not involve any ad hoc modifications of the unitary Schrödinger equation. It does, however, depict decoherence as necessarily "intrinsic", because even if all physical influences from the environment can be reduced to a minimum, no system can ever be topologically shielded from its global context, the integration of which conditions its state in a measurable, and therefore empirically significant way. The crucial point is that the view we are suggesting describes the reduction or collapse of quantum states (i.e. decoherence) as a genuinely topological process that is nevertheless physically relevant.

2. The Consistent Histories Approach to Quantum Mechanics

As already mentioned, the consistent histories approach to quantum mechanics aims at a description of closed quantum systems that can account for the emergence of classicality without reference to measuring devices or observers, thereby avoiding the problems of the standard Copenhagen interpretation with its notorious allusion to the collapse postulate. In terms of its formalism, the approach tries to assign probabilities to histories of closed systems, i.e. to alternatives of sequences of quantum events. It does so with a smaller set of theoretic postulates than standard quantum mechanics. Closed quantum systems are described by the common vocabulary of Hilbert spaces, vectors, and operators together with a formula for the ascription of probabilities to histories and an interpretation rule. One of its most striking features is that the approach does not assume a system-environment split by replacing the role that measurement events play in the standard interpretation by the importance of consistency criteria that describe the conditions under which classical reasoning might be applied to physical systems (see Halliwell 1995). Therefore the consistent histories approach, alongside pilot-wave theories and spontaneous collapse models, can be considered as a measurement-free as well as observer-free formulation of quantum mechanics.

However, the criteria of consistency that have been suggested so far in the context of the consistent histories approach fail to be sufficient for the exclusion of coarse-grained sequences of non-classical events. We argue that this is due to the fact that the difference between the algebraic structure of global quantum observables and the algebraic structure of local quasi-classical observables at coarse-grained scales is not taken into account in the consistent histories approach. We believe that this deficiency can be removed by topologizing the difference between these two types of observables in terms of a local-to-global sheaf-theoretic construction. Before turning to the details of the topological approach, let us take a look at the fundamental features and the problems related to consistent histories (readers familiar with the approach might prefer to proceed to the next section).

Imagine an electron visiting the Louvre in Paris. One possible and very simple report of the electron's visit could be the following: "At 9:12, the electron is in the Antiques Room; at 11:30 its velocity is between 2.3 and 4.5 km/h; and at 12:47 it is in the Corot Room." (This example is taken from Omnès 2002, p. 177.) A sequence of this kind gives a history of the electron. At certain moments in time, the value of a particular physical quantity is attributed to it; or more specifically, at each of the three chosen time points, a certain physical property of the electron, in the above case its position and its velocity, is said to be in a certain range of values.

To be sure, one could specify a practically infinite number of such possible histories describing the electron in the Louvre. Some of them contradict each other, and some are more likely than others. The consistent histories approach to quantum mechanics takes up the task of ascribing probabilities to different possible histories involving quantum objects such as electrons. How is this done formally?

The approach begins with the assumption of a discrete partial ordering of time points:

$$t_1 < t_2 < ... < t_n \tag{1}$$

Then, for each time point t_k a specific observable O_k is specified, the spectrum of which is partitioned into mutually exclusive domains $\Delta_i^{(k)}$ covering the whole spectrum of O_k such that

$$\Delta_i^{(k)} \cap \Delta_j^{(k)} = \varnothing \quad \text{for} \quad i \neq j \tag{2}$$

For each time point, there exists an exhaustive set of mutually orthogonal projection operators $\{\hat{E}_i^{(k)}\}|1 \leqslant i \leqslant n$. A projection operator $\hat{E}_i^{(k)}$ assigns a value within the domain $\Delta_i^{(k)}$ to O_k. Because any two domains in the spectrum of an observable O_k do not overlap and because all domains jointly cover the whole spectrum for any time point t_k, the projection operators obey the following equations:

$$\hat{E}_i^{(k)} \cdot \hat{E}_j^{(k)} = 0, \tag{3}$$

and

$$\sum_i \hat{E}_i^{(k)} = 1. \tag{4}$$

i.e. they are mutually exclusive and jointly exhaustive. A possible maximally fine-grained history H_α is defined by the sequence of times $t_1 < t_2 < ... < t_n$ and the choice of a projection operator from the set $\{\hat{E}_i^{(k)}\}$ for each time point:

$$\hat{E}_{\alpha_1}^{(1)}, \hat{E}_{\alpha_2}^{(2)}, ..., \hat{E}_{\alpha_n}^{(n)}. \tag{5}$$

The set of all possible histories forms a family of histories. A family of histories consisting in sequences of projection operators with the above properties is called a "Griffiths family" (Omnès 1988). One can define a specific operator, called "class operator", for each history in the family as the product of its mutually orthogonal projection operators belonging to the family:

$$\hat{C}_\alpha = \hat{E}_{\alpha_n}^{(n)} ... \hat{E}_{\alpha_2}^{(2)} \cdot \hat{E}_{\alpha_1}^{(1)}, \tag{6}$$

and it can be easily shown that the history operators of histories in a Griffiths family sum to unity:

$$\sum_\alpha \hat{C}_\alpha = 1. \tag{7}$$

Maximally fine-grained histories can be grouped into coarse-grained set of histories, which assign to each time point a linear combination of the original projection operators:

$$\hat{A}_{\beta_i}^k = \sum_{\alpha_i} \pi_{\alpha_i}^k \hat{E}_{\alpha_i}^{(k)} \quad \text{with} \quad \pi_{\alpha_i}^k \in \{1, 0\}. \tag{8}$$

One way of assigning probabilities to measurement outcomes in quantum mechanics makes use of the trace of the matrix that is obtained through the consideration of a projection operator and the initial state of the system represented by a density matrix $\rho(t_0)$:

$$p(i,t) = \mathrm{Tr}[\hat{E}_i^\dagger(t)\rho(t_0)\hat{E}_i(t)]. \tag{9}$$

Accordingly, the decoherent histories approach tries to attribute probabilities to histories using the trace rule in (9) and the class operator of the respective history:

$$p(\mathrm{H}_\alpha) = \mathrm{Tr}[\hat{C}_\alpha^\dagger \rho \hat{C}_\alpha]. \tag{10}$$

where ρ is the initial density matrix. It is important to realize that the probabilities thus obtained do not satisfy all the axioms of probability theory, and for that reason they are referred to as a candidate probabilities. More precisely they do not satisfy the requirement of additivity on disjoint regions of sample space. For each set of histories, one may construct coarser-grained histories by grouping the histories together according to equation (7). The additivity requirement is then that the probabilities for each coarser-grained history should be the sum of the probabilities of its constituent finer-grained histories. Quantum-mechanical interference generally prevents this requirement from being satisfied, such that:

$$p(\mathrm{H}_{\alpha\wedge\beta}) \neq p(\mathrm{H}_\alpha) + p(\mathrm{H}_\beta). \tag{11}$$

There are, however, certain types of histories for which interference is negligible, such that the candidate probabilities for histories do satisfy the sum rules. Consider a coarse-grained history arising from the combination of two fine-grained histories:

$$\mathrm{H}_{\alpha\wedge\beta} = \{\hat{E}^{(1)}_{\alpha_1} + \hat{E}^{(1)}_{\beta_1}, \hat{E}^{(2)}_{\alpha_2} + \hat{E}^{(2)}_{\beta_2}, ..., \hat{E}^{(n)}_{\alpha_n} + \hat{E}^{(n)}_{\beta_n}\}. \tag{12}$$

We take the sum of two projection operators $\hat{E}^{(k)}_{\alpha_k} + \hat{E}^{(k)}_{\beta_k}$ to mean that the system was at time k in a range described by the union of the two operators. It can be shown that probability of a history containing the joint projection operator can be calculated from the sum of the probabilities of the two histories containing the individual operators if the real part of the following complex-valued functional between the two histories α and β

$$\mathrm{D}(\alpha,\beta) = \mathrm{Tr}[\hat{C}_\alpha \rho \hat{C}_\beta] = \mathrm{Tr}[\hat{E}^{(n)}_{\alpha_n} ... \hat{E}^{(1)}_{\alpha_1} \rho \hat{E}^{(1)}_{\beta_1} \hat{E}^{(n)}_{\beta_n}] \tag{13}$$

vanishes, that is if $\mathrm{Re}[\mathrm{D}(\alpha,\beta)] = 0$, for all distinct pairs of histories α, β (see Griffiths 1984). Intuitively speaking, this functional measures the amount of interference between pairs of histories. The corresponding families of histories are said to be consistent, or weakly decoherent. The consistency condition is typically satisfied only for coarse-grained histories.

Since the consistency condition is satisfied in the case of absence of interference between two histories, it was also supposed to account for the emergence of classicality from underlying quantum processes. If this were the case, then the condition would provide an explication for the emergence of classical behavior in terms of an intrinsic mechanism, i.e. without the influence of an external agent such as the environment of a physical system, as it is done in approaches that try to account for the emergence of classicality in terms of environmentally induced decoherence. However, there exist many different families of consistent histories depending on how the coarse-graining is done. Those are equivalent in the sense that there is no privileged family of consistent histories. Yet, what is even more problematic, is the fact that due to interference effects, there might exist certain histories that exhibit highly non-classical behavior within one family of consistent histories in the sense that they correspond to highly non-classical series of events. Griffiths (1984), p. 233 writes: ”Such histories do exist, and the consistent history approach will assign probabilities to consistent families of ’grotesque’ histories if that is what interests the theoretician. The point we wish to make is not that grotesque events are somehow ruled out by the consistent history approach (obviously they are not), but simply that they are not an essential part of interpreting what happens in an ”ordinary” consistent history.” Also, it has been shown that is in principle possible to form infinitely many families of histories which imply *future* non-claissical behavior (Dowker and Kent 1995). So the approach has very week predictive power.

Therefore, the consistency condition, although it guarantees the applicability of the classical rules of probability, cannot assume the status of a sufficient criterion for classicality. Because of this deficiency, many authors have tried to augment the consistency criterion by appeal to interaction with the environment and thereby by a decomposition of the closed system (i.e. the universe) into subsystems in order to account for a selection of quasi-classical histories (Albrecht 1992, 1993; Anastopoulos 1996; Dowker and Halliwell 1992; Finkelstein 1993; Gell-Mann and Hartle 1993; Paz and Zurek 1993; Twamley 1993; Zurek 1993; all references taken from Schlosshauer 2005, p. 1300). Unfortunately, this runs against the original aim of the consistent histories approach, which was developed precisely to avoid the decomposition problem by taking a closed system perspective.

3. Topologizing Decoherence

We claim that the problems related to the consistency criterion derive the fact that the difference between the algebraic structure of quantum observables and the algebraic structure of quasi-classical observables at suitable coarse-grained scales is essentially ignored. The crucial point here is simply that the former are typically non-Boolean whereas the latter are always structurally Boolean. We also claim that the problems can be overcome by an appropriate topological treatment of this difference, and that this can be done without the introduction of a system-environment split. The crucial difference is not the one between system and some environment but rather the topological difference between global (quantum) and local (quasi-classical) frames of reference.

The relevance of reference frames in quantum mechanics has recently been been highlighted by Brown (2014). Brown uses the concept of a *quantum measurement frame* to exemplify a crucial feature of Bohr’s concept of complementarity, namely that the fundamental constituents of reality are "[...] processes of change in systems, not isolated systems whose behavior depends on non-relational *states*." (Brown 2014, p. 5). So Brown conceives of quantum frames not merely as space-time reference frames, but as defined by relations between a quantum system and what he calls the "exosystem", i.e. the measurement context and the remaining environment. This is in line with Rovelli's relational approach to quantum mechanics according to which "quantum mechanics is a theory about the physical description of physical systems relative to other systems, an this is a complete description of the world." (Rovelli 1996, p. 1650; quotation taken from Brown 2014, p. 8). In what follows we put forward a framework in which the difference between the classical nature of local frames of reference (Brown's quantum measurement frames) and the non-classical character of global quantum descriptions (Rovelli's complete description of the world) is taken into account. What is more, in opposition to Brown and the relationalist approach, who consider global quantum descriptions merely in terms of maximal collections of local frames, we formulate precise topological conditions that allow for the transition between local and global physical descriptions, and we explicate how this transition relates to the process of decoherence.

The key mathematical advantage of the topological approach, which uses the modern technical tools of category theory and sheaf theory for its precise formulation, is that the phenomenon of decoherence can be understood as a process of topological localization of a global quantum algebra of observables restricted to a particular local commutative algebra of quasi-classical observables associated with some macroscopic resolution scale.[3]

In order to make sense of the difference between the algebraic structure of quantum observables and the algebraic structure of quasi-classical observables, it is important to acknowledge that the global lattice of quantum events corresponds to a non-Boolean algebra. Put in general terms, a non-Boolean algebra is a lattice in which either the law of excluded middle or the law of non-contradiction do not hold. The non-Boolean lattice of quantum events is ortho-complemented and the lattice operations of disjunction and conjunction do not distribute over each other, or, equivalently, it can a be axiomatized as a partial Boolean lattice.[4]

In opposition to that, the algebra associated with local contexts (for instance local measurement contexts) are Boolean. This is reflected in the fact that the corresponding observables can be spectrally resolved into a set of mutually exclusive and jointly exhaustive projections. An it is well known that it is impossible to recover the information contained in the global algebra of quantum observables using a single Boolean context globally. This is basically the essence of the famous Kochen-Specker-Theorem according to which there are no simultaneous true/false evaluations of all the propositions in the

[3]A more detailed discussion of the topological approach to decoherence can be found in Epperson and Zafiris 2013, pp. 334-345.

[4]In the original formulation of Quantum Logic by Birchhoff and Von Neumann this fact is expressed by the identification of propositions with projection operators on a complex Hilbert space. See also Bub 1997.

quantum lattice (Kochen and Specker 1967). However, the Kochen-Specker-theorem does not rule out true/false evaluations with respect to Boolean sectors within the lattice. This means that the information contained in the global quantum lattice, although it can never be accessed via a unique global true/false evaluation, can nevertheless be approximated inductively by employing appropriate families of compatible or partially compatible Boolean frames.

Mathematically this move necessitates the introduction of variable Boolean algebras, acting as local frames, which are capable of collectively covering the global quantum algebra completely; what we are confronted with here, is a topological problem that consists in the task of covering a global algebraic structure by means of appropriate local elements. More precisely, this problem pertains to algebraic topology since in the case described above, the local elements as well as the global structure are algebras; Boolean algebras in the case of the local elements and a non-Boolean algebras in the case of the global algebra of quantum observables. It turns out that the appropriate technical tools to approach this problem are provided by sheaf theory, the essence of which is the approximation of a global structure by the topological interconnection of well understood local structures.[5]

In order to understand the transition of quantum algebras from the well known set-theoretic approach of the Hilbert Space formalism to the sheaf-theoretic approach, it is necessary to explain briefly the conceptual essence behind the Stone and Gelfand topological representation theorems. These theorems state that an arbitrary commutative algebra of observables $\mathcal{A}$ of a certain type (for instance a Boolean algebra of observables in Stone's case) may be represented as the algebra of continuous $\mathbb{F}$-valued functions, where $\mathbb{F}$ is a particular standard algebra (e.g. the algebra of real numbers or complex numbers or true/false logical values). This algebra of continuous functions is defined on a topological space associated with it. For example, the Stone representation theorem uses as the standard algebra is the two-valued Boolean algebra $\{0, 1\}$ and makes use of the discrete topology. The theorem is proved by the fact that a continuous function from the topological space X equipped with a discrete topology to $\{0, 1\}$, that is $f : X \to \{0, 1\}$ is completely determined by the clopen subset $f^{-1}(1)$ of X.

The extension of this topological representation approach to larger classes of algebras of observables turns out to be more complex in the sense that a general algebra of observables $\mathcal{A}$ cannot be constructed from a single standard algebra like the two-valued Boolean algebra $\{0, 1\}$ in Stone's case. An algebra of quantum observables falls into this category. But, as was already mentioned, the Kochen-Specker theorem does not preclude the local representation of the global information content by employing an appropriate multitude of compatible Boolean frames at different overlapping measurement scales. As already noted, this necessitates the introduction of continuously variable commutative algebras acting as local topological frames, which collectively cover the global

[5]For an introduction to algebraic topology and sheaf theory see Eisenbud 1995; Gelfand and Manin 2003; MacLane and Moerdijk 1992; MacLane 1998.

algebra completely. These local variable algebras form a so-called presheaf of algebras. The mathematical development of this topological schema is expressed suitably in the language of category theory and is based on the notion of a representation functor $\mathbf{\Upsilon}$ of the category of quantum algebras $\mathcal{L}$ into the category of presheaves of Boolean algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, defined by:

$$\mathbf{\Upsilon} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}} \tag{14}$$

The functor of Boolean reference frames $\mathbf{\Upsilon}(L)$ of a quantum algebra L in $\mathcal{L}$ is defined as the image of the representation functor, evaluated at L, into the category of presheaves of Boolean algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$, that is:

$$\mathbf{\Upsilon}(L) := \mathbf{\Upsilon}_L : \mathcal{B}^{op} \to \mathbf{Sets} \tag{15}$$

Note that the representation functor of $\mathcal{L}$, is completely determined by the action of the functor of Boolean reference frames for each quantum event algebra L in $\mathcal{L}$, on the objects and arrows of the category of Boolean event algebras $\mathcal{B}$, specified as follows: Its action on an object B in $\mathcal{B}$ is given by

$$\mathbf{\Upsilon}(L)(B) := \mathbf{\Upsilon}_L(B) = Hom_{\mathcal{L}}(B, L) \tag{16}$$

where $Hom_{\mathcal{L}}(B, L)$ is the set of morphisms from Boolean algebras B to a quantum event algebra L taken in the category $\mathcal{L}$. The action on a morphism $D \to B$ in $\mathcal{B}$, for $v : B \to L$ is given by

$$\mathbf{\Upsilon}(L)(x) : Hom_{\mathcal{L}}(B, L) \to Hom_{\mathcal{L}}(D, L) \tag{17}$$

$$\mathbf{\Upsilon}(L)(x)(v) = v \circ x \tag{18}$$

The fact that the functor $\mathbf{\Upsilon}_L$ of Boolean frames of L becomes a sheaf, means that there exists a gluing isomorphism of Boolean reference frames on their pullback over L (categorical overlap), defined as follows:

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(B \times_L \acute{B}) \to \psi_{B\acute{B}}(B \times_L \acute{B}) \tag{19}$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ {\psi_{\acute{B}B}}^{-1} \tag{20}$$

where $\psi_{\acute{B}B} : B \times_L \acute{B} \to \acute{B}$ and $\psi_{B\acute{B}} : B \times_L \acute{B} \to B$. The gluing isomorphism $\Omega_{B,\acute{B}}$ means that the Boolean reference frames of L, that is $\psi_{\acute{B}B}(B \times_L \acute{B})$ and $\psi_{B\acute{B}}(B \times_L \acute{B})$ cover the same observable information content of a quantum algebra L in a compatible way. A prove for the existence of the above isomorphism giving the transition functions from a Boolean frame B to a Boolean frame $\acute{B}$ can be found in Zafiris/Karakostas [6]

[6] A prove for the existence of this isomorphism giving the transition functions

In order to perform the transition from preshef to sheaf formally, we need to define an appropriate Grothendieck topology $\mathbf{J}$ on the base category of Boolean contexts $\mathcal{B}$, such that: [i]. The Boolean reference frames acquire the semantics of local frames with respect to that Grothendieck topology $\mathbf{J}$ on $\mathcal{B}$, and [ii]. The functor of Boolean frames for a quantum algebra L in $\mathcal{L}$ becomes a sheaf on the site $(\mathcal{B}, \mathbf{J})$ for that $\mathbf{J}$.[7] The site is the category of Boolean algebras equipped with an appropriate topology $\mathbf{J}$. This topology is defined in terms of covering sieves of Boolean frames, defined as follows: A B-sieve S on a Boolean reference context B in $\mathcal{B}$ is called a covering sieve of B, if all the Boolean homomorphisms $s : C \to B$ belonging to the sieve S, taken together, form an epimorphic family in $\mathcal{L}$. This requirement may be, equivalently, expressed in terms of a morphism:

$$G_S : \coprod_{(s:C \to B) \in S} C \to B \tag{21}$$

being an epimorphism in $\mathcal{L}$. It can be proved that the specification of covering sieves on Boolean contexts B in $\mathcal{B}$, in terms of epimorphic families of arrows in $\mathcal{L}$, does indeed, define a Grothendieck topology $\mathbf{J}$ on $\mathcal{B}$ (Zafiris 2004, 2006a).

Thus, the function $\mathbf{J}$, which assigns to each Boolean reference context B in $\mathcal{B}$, a collection $\mathbf{J}(B)$ of covering B-sieves, being epimorphic families of arrows in $\mathcal{L}$, constitutes a Grothendieck topology $\mathbf{J}$ on $\mathcal{B}$. Then, we can show that the functor of Boolean reference frames $\mathbf{\Upsilon}_L$ is a sheaf for the Grothendieck topology $\mathbf{J}$, defined by covering sieves of epimorphic families on the category of Boolean reference contexts, as above. Hence, the Boolean reference frames $B \to L$ acquire the semantics of local Boolean frames with respect to the Grothendieck topology $\mathbf{J}$, defined in terms of covering sieves of epimorphic families of arrows in $\mathcal{L}$, interpreted as Boolean localization systems.

It is instructive to emphasize that the concept of a categorical topology of the form $\mathbf{J}$ epitomizes the meaning of a localization scheme of a quantum observable algebra. The covering sieves are interpreted as generalized measures of topological localization of quantum observables at coarse-grained levels. The structure of a covering sieve obeys the Boolean structural rule. We can also think of the Boolean structural rule as a filter providing the consistency conditions according to which coarse-graining should be realized. The notion of a site generalizes the notion of a topological state space in categorical terms. In this way it provides the relational topological background for coarse-graining the global information contained in an algebra of quantum observables through localizing Boolean algebraic frames of quasi-classical observables at various resolution scales. Thus, from the physical point of view, covering sieves of a quantum algebra by local Boolean reference frames are interpreted as Boolean localization systems of a quantum algebra.

It is useful to think of covering sieves of a quantum algebra L as subfunctors of the functor of Boolean frames of L. These subfunctors are partially ordered by inclusion. The partial order relation among covering sieves of a quantum algebra L is interpreted

[7] For details of the definition of a Grothendieck topology see: Zafiris (2004).

as the operation of coarse-graining among Boolean localization systems. It should be emphasized that it is via these Boolean localization systems that a semantical transition from the set-theoretic to the topological sheaf-theoretic level becomes possible. This is justified by the following theorem: A covering sieve of a quantum algebra L by local Boolean reference frames, that is a Boolean localization system of L, induces an isomorphism:

$$L \cong \mathbf{Colim}\{\int(\mathbf{\Upsilon}_L, \mathcal{B}) \to \mathcal{B}\} \tag{22}$$

where, $\mathbf{\Upsilon}_L$ denotes a sheaf of Boolean frames of L on the site $(\mathcal{B}, \mathbf{J})$, for $\mathbf{J}$ the Grothendieck topology of epimorphic families, $\int(\mathbf{\Upsilon}_L, \mathcal{B})$ is the associated fibration induced by the category of Boolean frames of L, and **Colim** denotes the inductive limit operation (see Zafiris 2006a, p. 1501).

Thus, a quantum algebra L is represented isomorphically by the inductive limit taken in the category of local Boolean reference frames of the sheaf $\mathbf{\Upsilon}_L$ corresponding to a Boolean localization system. In a nutshell this implies that quantum algebras are not specified globally by their measured values, but by equivalence classes of compatible Boolean algebras forming an inductive limit in a sheaf of quasi-classical coarse-grained Boolean algebras covering the former at various resolution scales.

The basic idea of the topological modeling approach to coarse-graining induced decoherence can be summarized as follows: In spite of the fact that quantum systems obeys the laws of quantum logic and quantum probability, they can be described at macroscopic levels in terms of local Boolean sub-algebras of a global non-Boolean quantum algebra. These coarse-grained descriptions can be formulated for example in terms of macroscopic hydrodynamic variables or any given quasi-classical observables. This guarantees that the attribution of properties at these coarse-grained levels takes place locally according to the laws of Boolean logic and classical probability theory. Only in this case, then, is it meaningful to infer predictions about the behavior of the localized quantum system, via its local macroscopic reduction with respect to some local Boolean context.

Essentially this means that decoherence can be seen as restriction or reduction of a quantum system with respect to a local Boolean contexts, or more generally with respect to a Boolean localization system, as the result of coarse-graining subsumed by the function of covering sieves. The crucial point is that according to this framework, decoherence makes sense only as an intrinsic, locally conceived reduction of the global-level description of a quantum system.

Recent experimental work seems to verify this theoretical perspective. More specifically, within the context of environment-induced decoherence, it was assumed until now, that the environment modeled by means of a oscillator-bath acts externally to a quantum system via an appropriate physical interaction; most important this interaction has been conventionally presumed to extend globally. On the contrary, recent experiments (e.g qubits in a nuclear spin bath) suggest that decoherence is not necessarily connected to the dissipation of energy (see Stamp 2006; Bertaina et al. 2008). In the perspective of the topological approach to decoherence presented here, the environment can be thought of as the environment of topologically contextualized relations which nevertheless has physical relevance in the sense that it may trigger intrinsic decoherence without dissipation of energy. The ontological implication of this phenomenon is captured by the thesis that all causal relationships presuppose underlying topological relationships, and that topological relationships by themselves do have physical significance.

It should be emphasized that the topological approach not only models decoherence as localization via coarse-graining sieves, but also allows the conceptualization and modeling of recoherence from the local macroscopic to the global microscopic level via the gluing of locally compatible observables in the form of equivalence classes, technically called germs. It is also important to note that the global information of the quantum lattice is always preserved in the process of decoherence. So globally, coherence is retained. However, since the global quantum information necessarily has to be accessed via a local Boolean context or rather via equivalence classes of compatible Boolean contexts, it can never be captured directly; or, to put it other terms, we can never directly observe coherence, except indirectly and under very special circumstances such as the interference pattern in a double-slit experiment.

4. Conclusions

The application of the sheaf theoretic topological representation methodology to the case of a quantum algebra of observables sheds new light on the process of coarse-graining induced decoherence and the emergence of classical behavior. That decoherence might be fundamentally linked to coarse-graining has already been claimed (see for instance Castagnino et al. 2007). We believe, however, that an adequate *topological* understanding of the process of coarse-graining is crucial.

The topological compatibility relation between local Boolean frames of a global quantum algebra forces a strict criterion of consistency in the process of coarse-graining as a refinement of how it is currently implemented in the context of the consistent histories approach. More concretely, as we have already mentioned, the major problem of this approach is that the criterion of consistency of an arbitrary set of histories is not correlated with the requirement of classical behavior of a large class of observables. This problem is due to the fact that the process of coarse-graining as it is currently implemented does not have to satisfy any constraint; it is just a summation over projection operators which describes the transition from a fine-grained description to some coarse-grained description. We claim that an appropriate compatibility relation has to be imposed on the process of coarse-graining projection operators in order to obtain meaningful probabilities from the decoherence functional, and thus to explain the emergence of classical behavior at suitable coarse-grained macroscopic scales. The Boolean localization scheme implies that the coarse-graining process should respect the local Boolean structural rule as we pass from a local Boolean context to another; that is it should respect the morphology of internal relation between local Boolean frames covering a global quantum algebra. In order to express this formally we need a compatibility condition which is given precisely by the transition morphisms from a local Boolean frame to another local Boolean frame as prescribed by (19) and (20).

For example, referring to the same observable (e.g. position) the proper coarse-graining procedure corresponds to a nesting of local Boolean frames by means of a covering sieve, such that there exists compatibility on their overlaps preserving the local Boolean structural rule. So a detection of an electron in a region of space, say in the Corot Room, has to be compatible with a coarse-grained description of the electron being in the Louvre or in Paris. In case we consider *different* observables and apply the coarse-graining process properly, we require only *partial* compatibility on their overlapping Boolean frames (for instance, partial Boolean compatibility of coarse-grained position and momentum up to the limit of Heisenberg uncertainty relations) which is given by the topological gluing isomorphism (19).

It should also be emphasized that the formation of the inductive limit (22) is inherently linked to measurement procedures. It requires the formation of a set of equivalence classes (Boolean germs) of partially compatible Boolean measurement frames on all possible overlaps, glued together according to the transitions functions. If coarse-graining of global information is done according to the idea of gluing on overlaps such that the local Boolean structure rule is preserved, then it becomes apparent that this approach to coarse-graining resolves the problem associated with coarse-graining in the consistent histories approach, precisely because the Boolean structure is always locally preserved and non-classical descriptions are ruled out.

At this point, the question might arise what the physical meaning of the Boolean structural rule is. As we have seen, the Boolean structural rule is used to filter out non-classical sets of coarse-grained histories. But in what sense does this rule represent something physical rather than just a consequence of the experimenter's choice of appropriate local measurement frames? Or, to put it in other words, what that makes the proposed filter the *correct* or objective one, which can then serve as the criterion for consistence?

To address this question it is important to recognize the fact that the proposed approach rests on the presupposition that whenever we do physics, i.e. whenever we measure things, we have to choose a particular Boolean frame.That is to say we have to make a choice with respect to the observables that we want to use to probe the physical reality. In this sense, our approach takes into account the contextually of measurements and, so to speak, of real physics. Measurements necessarily take place in the classical regime and every single measurement requires the choice of a particular Boolean frame of reference. However, this should not lead to the misguided conclusion, that a single Boolean frame can be extended in order to obtain a global universal Boolean frame of reference. There is no such a thing. Rather, the task consists in finding a way of *objectively relating different local Boolean frames.* The Boolean structural rule serves this purpose by interrelating consistently all local Boolean frames in a covering sieve through the transition functions. An imaginary non-classical observer who lives in the quantum realm and would be forced to start his description in quantum algebraic terms. However, also this non-classical observer would have to use the proposed rule to descend from his global non-localized description to quasi-classical descriptions. Therefore, the objectivity of the Boolean structural rule is given by the fact that it provides a precondition for acquiring an invariant description for the transition from the local to the global as well as the other way around. In the formalism, the invariant object is represented by the sheaf of observable information.

This also sheds light on another unrecognized problem of the consistent histories approach, namely the presupposition of a partial order in terms of time points without justification of how this order is obtained in the first place. Literally all proponents of the approach begin their formalism with this arbitrarily given ordering of events.[8] By explicitly considering the local Boolean contexts of events and in particular the way that these contexts are glued together, our sheaf-theoretic approach *derives* the temporal order instead of merely presupposing it. It is of crucial importance, however, that the temporal order supervenes on underlying logical order of Boolean contexts as it is realized in a sieve. In consequence, we may think of the kinematical temporal order that the consistent histories approach presupposes, as derivative of the underlying logical order of local Boolean contexts in a sieve of a quantum event algebra.[9] A history then, from the topological perspective, is understood as a scheme of internally related local Boolean contexts. In the formalism of the proposed approach, histories are depicted in terms of covering sieves. Accordingly, the concept of a history subsumes not only a sequence of events but also the Boolean frames with respect to which they are contextualized.

With respect to the infamous measurement problem, our approach suggests the following: We can phrase the problem as the question why a physical systems in a coherent state evolves deterministically according to the unitary Schroedinger equation and why this description breaks down as soon as a measurement is performed. So the dynamical laws of quantum mechanics do not govern measurement processes, because we always observe classical outcomes. So measurement problem can conceived as arising from the attempt to reconcile two incompatible kinds of dynamics: one that governs the unitary time evolution of quantum states and the other that describes measurements processes. From the perspective of this approach, this problem appears ill-conceived. We observe only classical outcomes, because every measurement process takes place with respect to a local Boolean frame. The question should rather be how we can use different local Boolean descriptions to obtain a quantum description of the same physical setup. The answer to this question is clearly stated in our approach: By applying the topological glueing conditions and by forming equivalence classes in the way proposed above. This implies that what is usually seen as a non-unitary dynamical evolution, is in fact not of a dynamical nature at all, but of a logical nature having to do with the conditions imposed locally by the choice of specific local Boolean frames.

[8]To mention just one example, Griffiths 2002, p. 111 clearly presupposes a *given* ordered series of time points for quantum histories.

[9]The general fact that the partial order of a quantum event algebra is derivable from the relations of local Boolean frames belonging to a covering sieve has been been proven in Zafiris 2006a.

Also, the question whether the world is fundamentally quantum of fundamentally classical disappears in this perspective. Global quantum descriptions and local classical descriptions are complementary.[10] Note however, that the concept of complementary at work here, is not identical with the traditional notion introduced by Bohr. Bohrian complementarity refers to different local frames, e.g. the complementarity of position and momentum measurements. In our perspective, complementary must be taken in the sense of *vertical* complementarity between the local and the global level of physical descriptions.[11] And, as we have suggested, the correct way of passing from one level to the other is to apply topological restrictions to coarse-graining, which, at the same time, provide a viable consistency criterion for the process of decoherence.

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[10] Recall that the consistent histories approach is often referred to as "Copenhagen done right".

[11] For a formal treatment of vertical complementarity see Zafiris 2006b.

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